A Study of the Rational Involutorial Transformations in Space Which Leave a Web of Sextic Surfaces Invariant

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A THESIS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL OF CORNELL UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

BY

JESSE OTTO OSBORN

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By Jesse O. Osborn.

I. Introduction.

1. Consider two spaces (each of three dimensions), (x) and (x'), so related that a general point P_1 of (x) corresponds to a unique point P' of (x'), while a general point P' of (x') corresponds to two points P_1 and P_2 of (x). The correspondence between two such spaces may be expressed in such a way that the planes of the double space (x') correspond to surfaces of some order n in the single space (x) and that three such surfaces intersect in but two variable points. An involution of order 2 in (x) arises through the interchange of the points of associated pairs as P_1 and P_2; that is, P_1 (or P_2) being given, then P' and hence P_2 (or P_1) can be found. This involution will be designated by I.

2. Since the planes of (x') have four linear homogeneous degrees of freedom, the corresponding surfaces in (x) must belong to a web. This web of surfaces will be called ϕ. Each surface of the web must pass through certain fixed elements which together absorb $n^3 - 2$ intersections (equivalence) and impose $\frac{(n+1)\ (n+2)\ (n+3)}{6} - 4$ conditions on the coefficients of the equation of the surface (postulation). These basis elements consist of curves, simple or multiple on ϕ, of isolated points of various multiplicities, and of points on the basis curves which have greater multiplicity on the surfaces ϕ than the respective curves have. Each of these basis elements and the system as a whole have determinable equivalence and postulation. Two equations are formed. In one the equivalence of all possible combinations of basis elements is equated to the necessary equivalence, and in the other the postulation of all possible combinations of basis elements is equated to the necessary postulation. It is assumed throughout that the basis elements are all independent or in other words that no basis element occurs as a consequence of others. Any solution of these equations, which leads to no geometrical contradiction, furnishes a set of basis elements for ϕ. In determining the equivalence and postulation of the various basis elements the following formulas will be useful.

(1) If a curve C_m is i_1, i_2 and i_3 fold on the three surfaces F_{n_1}, F_{n_2} and F_{n_3} respectively, it is equivalent to E_m points of intersection of the three surfaces, where (r being the rank of C_m)

$$E_m = m(i_2i_3n_1 + i_3i_1n_2 + i_1i_2n_3 - 2i_1i_2i_3) - ri_1i_2i_3.$$

(2) The equivalence of two curves C_m and $C_{m'}$ equals the sum of their separate equivalences minus the quantity $(i'_2i'_3i_1 + i'_3i'_1i_2 + i'_1i'_2i_3 - i'_1i'_2i'_3)s$, where s is the number of points in which C_m and $C_{m'}$ intersect and $i'_1 \leqq i_1$, $i'_2 \leqq i_2$, $i'_3 \lesseqgtr i_3$.

(3) The postulation of a curve C_m, which is i-fold on a surface F_n is

$$P_m = \frac{i(i+1)}{2}\left(m(n+2) - \frac{2i+1}{6}(r+2m)\right).$$

(4) If a point P_{ji} is j_1, j_2 and j_3 fold on the surfaces F_{n_1}, F_{n_2} and F_{n_3} respectively and lies on the curve C_m, then the added equivalence of P_{ji} is given by the formula

$$j_1j_2j_3 - (i_2i_3j_1 + i_3i_1j_2 + i_1i_2j_3 - 2i_1i_2i_3).$$

(5) The added postulation of P_{ji} is

$$\frac{j(j+1)(j+2)}{6} - \frac{i(i+1)}{2}\left(j+1-\frac{2i+1}{3}\right).$$

The order and genus of a basis curve i-fold on each surface ϕ are denoted by m_i and p_i respectively, (or where no ambiguity is possible simply by m and p). The number of points of an i-fold basis curve which are j-fold on ϕ is designated by β_{ji}. If $i = 0$, these are isolated basis points and if $j \leq i$ the symbol has no meaning. In the above formulas and in the equations of equivalence and postulation derived from them no account is taken of the possibility that a point at the intersection of two or more basis curves be of greater multiplicity on ϕ than would naturally follow from its being a point of the basis curves. If such account were taken the equations would become extremely complicated and probably few if any essentially new cases would arise.

The method here used was devised by Sharpe and Snyder.* By means

* F. R. Sharpe and V. Snyder, "Certain Types of Involutorial Space Transformations," *Transactions American Mathematical Society*, vol. 20 (1919), pp. 185-202, and vol. 21 (1920), pp. 52-78. Special cases had been considered earlier by Aschieri, "La trasformazione quadratica doppia di spazio, e la sue applicazione alla geometria dello spazio non euclideo," *Rendiconti del r. Istituto Lombardo*, ser. 2, vol. 14 (1881), pp. 673-683, and vol. 15 (1882) pp. 66-77, 147-154, 247-250; by Reye, "Ueber die Kum-

of it they have studied a large number of the involutorial correspondences that exist when the surfaces ϕ are of order less than six. In this paper the study is extended to the case where these surfaces are sextics. Whenever ϕ is mentioned in the succeeding pages, it refers to this web of sextic surfaces.

3. Besides basis elements any two surfaces ϕ meet in a variable curve $C\mu$ of order μ and genus π, image of a straight line in (x'). $\mu = 36 - \Sigma i^2 m_i$. Assuming the surfaces ϕ to be regular and the basis elements to be independent, π is given as one greater than the arithmetic genus of a general surface of the web. A basis curve C of ϕ has for image in (x') a surface S' of order equal to the number of points in which $C\mu$ meets C. Each point of C has for image in (x') a curve on S' whose order equals the multiplicity of C on ϕ. A basis point of ϕ has for image in (x') a surface of order equal to the multiplicity of the point on $C\mu$; each direction through the basis point corresponds to a curve on the image surface of order equal to the multiplicity of the basis point on ϕ.

The image in (x') of an arbitrary curve c of order m is a curve c' of order $6m$. For each intersection of c with a basis element k-fold on ϕ, the order of the image curve c' is reduced by k. If c meets enough basis elements of (x) to make $k = 6m$, its image is no longer a curve but simply a point. $C\mu$ meets so many basis elements that its image instead of being of order 6μ is only a line. There may be various curves in (x) that thus correspond to points in (x'), that is each point of such a curve corresponds to a direction through the image point. Such a curve is said to be parasitic.** These curves may be isolated in which case the number that are lines, conics, cubics, $\cdots$ will be designated by α_1, α_2, α_3, $\cdots$, or such a curve may generate a parasitic surface. Every isolated parasitic curve which has been found satisfies the following conditions (1) it is rational; (2) it adds 1 in postulation to the basis elements of ϕ; (3) it adds equivalence 2 to the basis elements of ϕ; (4) In I each point of it corresponds to the whole curve while the curve as a whole is self-corresponding. A parasitic curve which generates a surface may behave in various ways. In this paper will be considered only those cases where the parasitic curve either (a) satisfies the four conditions enu-

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mersche Configuration von sechzehn Punkten und sechzehn Ebenen, *Crelle's Journal*, vol. 86 (1879), pp. 209-212, and "Ueber Strahlensysteme zweiter Classe und die Kummersche Fläche vierter Ordnung mit sechzehn Knotenpunkten," *Crelle's Journal*, vol. 86 (1879), pp. 84-107; and particularly by R. De Paolis, "Le trasformazioni doppie dello spazio," *Memorie della r. Acc. dei Lincei*, ser. 4, vol. 1 (1885), pp. 576-608. The last memoir contains many general formulas.

** This property was first discussed by Schoute, "De la transformation conjuguée dans l'espace," *Association française pour l'avancement des sciences*, Congres de Reims (1880), and more in detail by de Paolis (l. c., p. 595). They use this designation.

merated above, or (b) satisfies (1) and (2) while (3′) it adds equivalence 1 to the basis elements of ϕ so that (4′) in I each point of it corresponds to the whole curve plus another point while the curve as a whole corresponds to this residual point.

In general a point P' of (x') corresponds to two points P_1 and P_2 of (x); for certain positions of P', P_1 and P_2 coincide. The totality of such positions of P' make up the surface of branch points L'; the corresponding points $P_1 \equiv P_2$ make up the surface of coincident points $\overline{K}$, that is $\overline{K}$ taken doubly is the complete image of L'. If a parasitic curve q is self-corresponding in I, then it must lie on $\overline{K}$ and its image Q' must be a point of L'.

The parasitic surfaces generated by curves satisfying (a) or (b) belong to one of two classes. If, in I, the parasitic curve which generates it is self-corresponding or corresponds to a point on the parasitic surface generated, that surface is self-corresponding in I and will be called D. A parasitic surface which is a fixed component of two (or three) linearly independent surfaces ϕ and is such that the pencil (or net) of residual components intersect in basis curves only is self-corresponding in I. All surfaces D that occur in this paper satisfy this condition. It follows that the image in (x') of a surface D is a line d' or a point D'. As D is self-corresponding in I, d' or D' must lie on L' and hence D is a component of $\overline{K}$. After all surfaces D have been taken from $\overline{K}$ the necessary number of times there remains the proper surface of coincidences K. If a parasitic surface is not self-corresponding in I it will be called R. All surfaces R which occur in this study are ruled. Each generator corresponds to a point in (x') and to a point in I; R as a whole corresponds to a curve r' or a point R' in (x') and to a curve r in I. The surface R has the same number of generators in common with K as r' has contacts with L'. These particular generators will be designated as Σt.

The correspondence between (x) and (x') was set up such that the planes of (x') correspond to the web of sextic surfaces through the basis elements of (x), or symbolically, $s'_1 \sim s_6 : \Sigma C_m + \Sigma P$. A plane s_1 of (x) corresponds to a surface of order μ in (x'). Each plane of (x) meets a parasitic curve q of order m in m points; it follows that s'_μ contains Q' m-fold. If Q' describes a curve (in which case q generates a parasitic surface F), then s'_μ contains this curve m-fold. The image in (x) of s'_μ is a surface of order 6μ consisting of the original plane s_1, of F taken m-fold and of a surface s_N. The web of surfaces s_N contains all the basis elements of ϕ to definite multiplicities, contains m-fold each isolated parasitic curve of order m and contains r and Σt if they exist.

4. The existence as well as the nature of an involution depends largely

on the value of π. The order of L' is given by Zeuthen's formula as $2(\pi+1)$; it follows that the order of $\bar{K}$ is $6(\pi+1)$. The multiplicity of each of the basis elements of ϕ on $\bar{K}$ is known. The jacobian J of web ϕ is of order 20. A curve k-fold on ϕ is $4k-1$ fold on J and a point k-fold on ϕ is $4k-2$ fold on J. The jacobian consists of K together with all parasitic surfaces. By a comparison of J and $\bar{K}$ for different values of π one can find what parasitic surfaces may occur. For convenience, at this point only, let C_a, C_b, C_c, $\cdots$ and A, B, C, $\cdots$ designate respectively simple, double, triple, $\cdots$ curves and points, basis on ϕ. Consider

$$\bar{K}_{6(\pi+1)} : C_a^{\pi+1} + C_b^{2(\pi+1)} + C_c^{3(\pi+1)} + \cdots + A^{\pi+1} + B^{2(\pi+1)} + C^{3(\pi+1)} + \cdots$$
$$J_{20} : C_a^{3} + C_b^{7} + C_c^{11} + \cdots + A^{2} + B^{6} + C^{10} + \cdots$$
$$F_{14-6\pi} : C_a^{2-\pi} + C_b^{5-2\pi} + C_c^{8-3\pi} + \cdots + A^{1-\pi} + B^{4-2\pi} + C^{7-3\pi} + \cdots$$

No non-reducible case enters for which π is zero. If π is 1 or 2 there must be a parasitic surface R and if π exceeds 2 there must be a parasitic surface D. For π equal to 2, ϕ may have no simple basis point for such would necessitate an untrue relation among the multiplicities of such a point on R_2, $\bar{K}$ and J. There can be no simple basis point of ϕ when π is equal to 3. In order that a D_4 exist it must be a component of (at least) two surfaces ϕ. Each basis point simple on ϕ is double on D_4. One of the quadric surfaces which with D_4 makes up a surface ϕ will contain this point while a web of quartic surfaces contains the remaining basis elements of ϕ. Thus any I that might exist can be associated with a web of quartic surfaces. If π equals or exceeds 4, $D_{6\pi-14}$ must be composite in such a way that each component satisfies the conditions which characterize a surface D. For π equal to 4 the only case that has been found to exist is where D_{10} consists of a double quintic surface. This necessitates that ϕ have no point of odd multiplicity and no curve of even multiplicity. No case has been found where π exceeds 4.

The existence of a surface D implies that there is a pencil (or net) of surfaces ψ_i, (such that $D\psi_i$ is a surface ϕ), which corresponds to a pencil (or bundle) of planes in (x'). If ψ_i is known then D is also known. In all cases the basis elements of ψ_i must satisfy the conditions (1) as to equivalence, (2) as to postulation, and (3) as to order of the variable intersection of two ψ_i. These conditions always suffice to characterize the sets of basis elements possible for ψ_i. As an example suppose D is of order 4 and corresponds to a line d'. In this case ψ_i is a pencil of quadric surfaces through certain basis elements of (x). The equivalence and postulation of these basis elements must each equal 8. Two of the quadric surfaces must intersect in basis curves whose orders total 4. A curve of order 4 and genus 1 is the only locus that satisfies these conditions. No set of basis elements is possible for

ψ_i such that D_4 may correspond to a point. As the need arises the criterion set forth in this paragraph will be used as a test for the existence of surfaces D.

When the image of D is $2i$-fold on L', D is an i-fold component of $\overline{K}$. According as D is a component of two or three surfaces ϕ, it is a simple or double component of J.

5. Besides parasitic surfaces there are a_1 lines, a_2 conics, a_3 cubics, $\cdots$ which meet basis elements of (x) in enough points to correspond to points in (x'). These curves are self-corresponding in I. It is possible to get various relations among the a's such that their values may be obtained. (1) from the correspondence between (x') and (x) one knows through what basis curves s_N goes; s_N also has each parasitic curve to multiplicity equal to the order of the curve. Suppose two s_N meet in basis curves whose combined orders, taking account of multiplicity and of r and Σt, equal M. The variable intersection of two s_N is of order N. It follows that $a_1 + 8a_2 + 27a_3 + 64a_4 + \cdots = N^2 - N - M$. (2) The multiplicities of the various basis curves on K are also known. Each parasitic curve is simple on K. Let the combined orders of the basis curves common to K and s_N be M'. When the order of K is N' its variable intersection with an s_N is of order N'. It follows that $a_1 + 4a_2 + 9a_3 + 16a_4 + \cdots = NN' - N' - M'$. (3) The plane s_1 and its image s_N meet in the curve $k_{N'}$, intersection of s_1 and $K_{N'}$, and a curve g of order $N - N'$. The basis and parasitic curves of ϕ pierce s_1 as points of $k_{N'}$ and $g_{N-N'}$. If a curve is m-fold on s_N and m'-fold on $K_{N'}$ then the points in which it pierces s_1 are m' and $m - m'$ fold on $k_{N'}$ and $g_{N-N'}$ respectively. The curves k and g meet in $N'(N - N')$ points which consist of the points in which the basis and parasitic curves pierce s_1 and of η other intersections. Suppose points on the basis curves account for M'' points. Then $2a_2 + 6a_3 + 12a_4 + \cdots + \eta = N'(N - N') - M''$. The number of branch points η in the $(1, 2)$ correspondence between g and its image curve g', the double curve of s'_μ, is given by Zeuthen's formula as $\eta = (2p_g - 2) - 2(2p_{g'} - 2)$. The genus of g designated as p_g is given as $\frac{(m-1)(m-2)}{2} - h$ where the order m and the number of double points h, points of basis or parasitic curves, are known. The order of g' may be found as the number of points in which it pierces a plane s'_1, or what is equivalent as one half the number of non-basis points in which g pierces the surfaces ϕ. The genus $p_{g'}$ of g' is obtained by finding the arithmetic genus of s'_μ by two methods and equating the results. In this equation $p_{g'}$ enters as the only unknown other than the numbers of parasitic curves. Thus η may be written in terms of the a's. (4) A parasitic curve of order m is an m-fold basis curve on the web s_N. One can calculate the postulation necessary to fix the web s_N and also

how much of this arises from the basis elements. Let the difference, a negative number, be designated by $-N''$. The postulation of the parasitic curves must equal $-N''$ or in equation form $a_2 + 4a_3 + 10a_4 + 20a_5 + \cdots = N''$.

Only zero or positive integral values of a are usable. If all correspondences are correct, however, there must be at least one such set of values for the a's which satisfies the above relations. There may be several sets. Each case offers some check that aids one in deciding which solution (or solutions) is correct. By known formulas * the number of fundamental lines may be calculated independently of the above relations; sometimes this may also be done for the number of parasitic conics or other parasitic curves.

6. In proceeding, a systematic and exhaustive search is made for the consistent sets of basis elements of ϕ. This is done by assuming a certain curve (or curves) or set of points to be basis and finding what other basis elements can be added in order that the combination satisfy the equations of equivalence and postulation and present no geometrical contradiction.

II. Discussion of Cases.

A. *Basis Elements Include a Quadruple Curve.*

7. It is impossible that a curve be of multiplicity greater than 4 on a web of sextics. If a curve is 4-fold on ϕ it cannot have order greater than 1. Suppose the basis elements include a 4-fold line C_1. In order that C_μ be non-composite it is necessary that there exists no other basis curve of multiplicity greater than 1. There can exist no basis point, which is not a point of C_1, of multiplicity greater than 2 for in such case the plane of C_1 and the point would be a fixed component of each ϕ. No solution results when one assumes that a point of the 4-fold line is 5-fold on ϕ. With these restrictions the equations of equivalence and postulation are

$$\begin{aligned} 8\beta_{20} + 4\beta_{21} + \beta_{10} &= 52 - 14m + 2p + 11s \\ 4\beta_{20} + 2\beta_{21} + \beta_{10} &= 29 - 6m + p + 4s \end{aligned} \qquad m \equiv m_1$$

The variable intersection C_μ is of order $\mu = 20 - m$ and has genus $\pi = 1$. A plane through C_1 meets each surface ϕ in a residual conic. These conics can go through not more than two fixed points, that is $s \geq m - 2$. Suppose three points are fixed. We can choose three others on the same plane such that a conic will not go through the six points in question. The images of the three points chosen determine a plane in (x'). The sextic surface, ϕ image of this plane, must go through the six points. The only way this can

* See Pascal's *Repertorium*, 1st edition, vol. 2, p. 231.

happen is for the plane through C_1 to be a component of this particular s_6. This would necessitate that the pencil of planes through C_1 correspond to a pencil of planes in (x'). Such is not the case here. Obviously s cannot exceed m. Three cases may arise according as s equals m, $m-1$, or $m-2$. If $s=m$, C_m must consist wholly of lines and conics skew to each other but coplanar with C_1. If $s=m-1$, C_m must either be rational or composite in such a way that one component is rational and the rest consist wholly of lines and conics skew to each other but coplanar with C_1. There is no restriction on C_m if $s=m-2$. The equations of equivalence and postulation have a large number of solutions. However, most of the cases that might result from these are excluded or reduced birationally by one of the following devices.

(a) Basis elements may occur as a consequence of others. An example of this is where $s=m-2$ and there is a double point. Through this double point two lines can be drawn which meet each ϕ in enough points to become basis. No such case is treated in this discussion.

(b) If there exist as many as two double points on ϕ, either isolated or on C_m, the sextic surfaces which generate the involution may be reduced by a $(2,3)$ birational transformation. The basis elements in (x) of this transformation are C_1 and three basis points, two of which must be double on ϕ.

(c) If C_m has for components at least three skew lines each meeting C_1 the web ϕ may be reduced birationally by a $(3,2)$ transformation. The basis elements in (x) of this transformation are C_1 taken double plus the three skew lines.

(d) If C_m has as a component a line meeting C_1 and there is a double basis point the web of sextic surfaces may be reduced by a $(2,2)$ birational transformation. The basis elements in (x) of this transformation are C_1, the line meeting C_1, and the double point.

The following are the only cases not eliminated by one of the above methods.

	m	p	s	β_{10}	μ	π
A	12	3	10	0	8	1
B	11	2	9	1	7	1
C	10	1	8	2	6	1
D	9	0	7	3	5	1
E	8	−1	6	4	4	1

For discussion the group will first be considered as a whole using s as parameter.

$$m=s+2,\quad \mu=18-s,\quad p=s-7,\quad \beta_{10}=10-s,\quad 6\leq s\leq 10;$$

$$L'_4 \sim K_{12} : C^8{}_1 + C^2{}_{s+2} + (10-s)P^2, \quad J_{20} \equiv K_{12} + R_8 : C^7{}_1 + C_{s+2}.$$

The surface R_8 is generated by the bisecants of C_{s+2} which meets C_1. The image of R_8 is a curve r'_{18-s}. The order of r' is given as the number of points in which it meets a plane s'_1 or, what is equivalent, as the number of lines in which the corresponding s_6 meets R_8. Any plane s_1 meets each generator of R_8; hence the image surfaces s'_{18-s} contain the curve r'. The surface s'_{18-s} has for image in (x) a surface of order $108 - 6s$ consisting of s_1, R_8 and $s_{99-6s} : C^{65-4s}_1 + C^{17-s}_{s+2} + (10-s)P^{18-s}$. The image of s_1 in I is the surface s_{99-6s}. The curve r' has for image in (x) R_8 plus the curve r of order $s^2 - 33s + 267$. The order of r is given as the number of points in which it meets s_1 or, what is equivalent, as the number of lines common to R_8 and s_{99-6s}. There are $36 - 2s$ lines Σt common to K_{12} and R_8; these correspond to the $36 - 2s$ contacts of r' with L'_4.

The first three of the devices explained in section 5 suffice to find the numbers of the various parasitic curves. The plane s_1 and its image s_{99-6s} intersect in k_{12} and g_{87-6s}. The image g' of g is of order $\frac{s^2 - 33s + 264.}{2}$

The genera of g and g' and the number of branch points η are respectively

$$p_g = \frac{3698 - 726s + 47s^2 - s^3}{2} - 3a_3 - 12a_4 - 30a_5 - 60a_6 \cdots$$

$$p_{g'} = \frac{7014 - 1411s + 93s^2 - 2s^3}{6} - 2a_3 - 8a_4 - 20a_5 - 40a_6 \cdots$$

$$\eta = \frac{s^3 - 45s^2 + 644s - 2928}{3} + 2a_3 + 8a_4 + 20a_5 + 40a_6 \cdots$$

The relations among the numbers of the various parasitic curves resulting from (1), (2) and (3) of section 5 are respectively

$$a_1 + 8a_2 + 27a_3 + 64a_4 + 125a_5 + 216a_6 + \cdots = 4596 - 848s + 51s^2 - s^3$$

$$a_1 + 4a_2 + 9a_3 + 16a_4 + 25a_5 + 36a_6 + \cdots = 552 - 68s + 2s^2$$

$$2a_2 + 8a_3 + 20a_4 + 40a_5 + 70a_6 + \cdots = \frac{4512 - 842s + 51s^2 - s^3}{3}$$

The number of parasitic lines a_1 is calculated independently to be $2(10-s)$, for two lines can be drawn through each basis point which will meet both C_1 and C_{s+2}. These lines are therefore parasitic. For this value of a_1 the above equations have an unique set of solutions. These solutions together with the numerical orders and genera of g and g' and the value of η are given in the following table.

	s	a_1	a_2	a_3	a_4	a_5	a_6	g	g'	p_g	$p_{g'}$	η
A	10	0	0	8	0	0	0	27	17	45	18	20
B	9	2	0	4	4	0	0	33	24	61	25	24
C	8	4	0	2	4	2	0	39	32	79	33	28
D	7	6	0	1	3	3	1	45	41	99	42	32
E	6	8	0	0	4	0	4	51	51	121	52	36

The image in (x') of the line C_1 is a surface S' of order $16-s$ which contains r'; the image in (x) of S'_{16-s} is R_8 plus the surface S_{88-6s} : $C_1^{58-4s} + C_{s+2}^{15-s} + r + \Sigma t + \Sigma a + (10-s)P^{16-s}$. The curve C_{s+2} corresponds to a surface T' of order $32-s$ containing C_1 double; T' in turn corresponds to $R^2{}_8$ plus the surface

$$T_{176-6s} : C_1^{114-4s} + C_{s+2}^{31-s} + r^2 + \Sigma t^2 + \Sigma a + (10-s)P^{32-s}.$$

Each basis point corresponds in (x') to a plane and in I to a sextic surface of web ϕ which has a double point at the original basis point.

The web of sextic surfaces having a quadruple line can be generalized to a web of n-tic surfaces having an $n-2$ fold line. The correspondence is

$$J_{4n-4} : C_1^{4n-9} + C^3{}_{s+2} \sim [K_{2n} : C_1^{2n-4} + C^2{}_{s+2}] + [R_{2n-4} : C_1^{2n-5} + C_{s+2}].$$
$$R_{2n-4} \sim r'_{5n-s-12} \sim R_{2n-4} + r_{20n^2-n(9s+94)+(s^2+21s+111)}.$$
$$s_1 \sim s'_{4n-s-6} : r' \sim s_1 + R + s_{4n^2-ns-8n+3} : C_1^{4n^2-6n-ns+2s+17} + C_{s+2}^{4n-s+3}.$$

The surface R_{2n-4} is generated by the bisecants of C_{s+2} which meets C_1.

B. *Basis Elements Include a Triple Curve.*

8. The order of a curve triple on a web of sextics cannot exceed 1, unless each surface of the web has a fixed component or the web as a whole is reducible. The truth of this is evident upon testing the various possibilities. Assume the basis elements of ϕ to include the triple line C_1. There can be no point, which is not a point of C_1, that is more than 3-fold on ϕ for such would necessitate that the plane of C_1 and this point be a component of each s_6. From the equations of equivalence and postulation it is found that no point of C_1 may be 4- or 5-fold on ϕ. The line C_1 together with three basis points whose combined multiplicities on ϕ equal at least 7 make up the basis system of a $(2,3)$ birational transformation by means of which the order of web ϕ may be reduced. This precludes the possibility of two or more triple points or of one triple point and two or more double points. In continuing there are two cases according as there is or is not a double curve among the basis elements of ϕ.

Suppose there is a double basis curve associated with the triple line C_1.

If the order of this double curve exceeds 1 all cases that arise are at once reducible. Assume there is a double line $\bar{C}_1$. If C_1 and $\bar{C}_1$ intersect a $(2, 2)$ birational transformation whose basis elements are these lines together with a point reduces the order of the surfaces ϕ. There can be no triple basis point which is not a point of $\bar{C}_1$ and no point of C_1 may be triple if there is a double basis point off this line, for in all such cases the above $(2, 3)$ birational transformation works a reduction in the order of the surfaces ϕ.

Suppose there is no double basis curve but that there is a simple basis curve C_m. If there is a 3-fold basis point then $m - s \leq 3$, for otherwise the plane of C_1 and the triple point would meet each ϕ in basis curves only. Under this condition the equations of equivalence and postulation have no solution.

9. Suppose the basis system of ϕ includes the triple line C_1 and the double line $\bar{C}_1$. Any plane through C_1 meets the surfaces ϕ in residual cubics. Two of these cubics meet in nine points four of which are accounted for at the point where $\bar{C}_1$ pierces the plane. Not more than two of the remaining five points can be fixed. Therefore $m - s_3 \leq 2$ where s_3 is the number of points in which the simple curve C_m meets the triple line. By the same argument it is shown that $m - s_2 \leq 2$, s_2 being the number of points common to C_m and $\bar{C}_1$. The equations of equivalence and postulation are

$$8\beta_{20} + 4\beta_{21} + \beta_{10} = 48 - 14m + 2p + 8s_3 + 5s_2$$
$$4\beta_{20} + 2\beta_{21} + \beta_{10} = 26 - 6m + p + 3s_3 + 2s_2.$$

A number of cases arise from the solutions of the above equations; these are differentiated from one another by the nature of C_m and the distribution of the basis points. The method of treatment is the same in all cases. The group will be treated as a whole using m as parameter.

$$L'_4 \sim K_{12} : C_1^6 + \bar{C}_1^4 + C_m^2 ;$$
$$J_{20} : C_1^{11} + \bar{C}_1^7 + C_m^3 \equiv K_{12} + R_8 : C_1^5 + \bar{C}_1^3 + C_m .$$
$$R_8 \sim r'_{27-m} \sim R_8 + r_{m^2-47m+529}.$$
$$s_1 \sim s'_{23-m} : r' \sim s_1 + R_8 + s_{129-6m} : C_1^{64-3m} + \bar{C}_1^{43-2m} + C_m^{22-m}.$$

10. Suppose the basis elements of ϕ are the triple line C_1, a simple curve C_m and basis points. There can be no basis point of multiplicity greater than 2. The equations of equivalence and postulation are

$$8\beta_{20} + 4\beta_{21} + \beta_{10} = 104 - 14m + 2p + 8s$$
$$4\beta_{20} + 2\beta_{21} + \beta_{10} = 45 - 6m + p + 3s.$$

A plane containing C_1 intersects the surface ϕ in residual cubic curves. Two of these curves intersect in nine points. These are points off C_1 where

C_m and C_μ pierce the plane. As C_μ is not rational (genus 4) it must meet the plane in at least two points which are not points of C_1. Therefore $m - s \leq 7$. From the above equations it is found that $2m - 2s - \beta_{10} = 14$. It follows that $m - s = 7$ and $\beta_{10} = 0$. Using s as parameter, $m = 7 + s$, $p = 3s - 3 + 4\beta_{20} + 2\beta_{21}$, $\mu = 20 - s$, $0 \leq s \leq 13$.

There exists a parasitic surface $D_5 : C^2{}_1 + C_m + \Sigma P^2$ which is self-corresponding in I and corresponds to a line d' in (x'). The pencil of planes through C_1 corresponds to the pencil of planes through d'. Designate a plane through C_1 as w and the corresponding plane through d' as w'. The lines of w correspond to cubic curves of genus 0 in w' and conversely. A point of d' corresponds to a curve of order $17 - s$ on D_5. Therefore d' is $17 - s$ fold on each surface s', image of a plane in (x). The surface of branch points L'_{10} has d' 6-fold. The correspondence may be expressed

$$L'_{10} : d'^6 \sim D_5^3 + K_{15} : C_1^9 + C^2_{s+7} + \Sigma P^4 ; \; J_{20} \equiv D_5 + K_{15}.$$
$$s_1 \sim s'_{20-s} : d'^{17-s} \sim s_1 + D_5^{17-s} + s_{34-s} : C_1^{26-s} + C^3_{s+7} + \Sigma P^6 .$$

The number of the various parasitic curves may be obtained by methods previously explained. A plane s_1 and its image s_{34-s} in I meet in k_{15} and g_{19-s}. The image g' of g is of order $28 - 2s$. The genera of g and g' and the number of branch points in the $(1, 2)$ correspondence between them are

$$p_g = (17 - s) - 3a_3 - 12a_4 - 30a_5 - 60a_6 \cdots$$
$$p_{g'} = 0 - 2a_3 - 8a_4 - 20a_5 - 40a_6 \cdots$$
$$\eta = (36 - s) + 2a_3 + 8a_4 + 20a_5 + 40a_6 \cdots$$

Consideration of the intersection of two s_{34-s}, of an s_{34-s} and K_{15}, and of k_{15} and g_{19-s} leads respectively to the equations

$$a_1 + 8a_2 + 27a_3 + 64a_4 + 125a_5 + 216a_6 + \cdots = 383 - 24s$$
$$a_1 + 4a_2 + 9a_3 + 16a_4 + 25a_5 + 36a_6 + \cdots = 219 - 12s$$
$$2a_2 + 8a_3 + 20a_4 + 40a_5 + 70a_6 + \cdots = 82 - 6s$$

The only solution of the above equations is

$$a_1 = 55 ; \qquad a_2 = 41 - 3s ; \qquad a_i = 0, \; i > 2.$$

A parasitic line either meets C_1 once and C_{s+7} three times or goes through a double basis point and meets both C_1 and C_{s+7}; thus the number a_1 may be calculated independently.

The line C_1 corresponds in (x') to a surface S'_{18-s} having d' $15 - s$ fold, and in I to a surface S_{33-s}. The curve C_{s+7} corresponds in (x') to a surface T'_{64-3s} having d' $57 - 3s$ fold, and in I to a surface T_{99-3s}.

C. *Basis Elements Include a Double Curve.*

11. Suppose the basis elements of ϕ consist of a double curve, a simple

curve and points of various multiplicities. The order of the double curve cannot exceed 8. If the order is 8, the genus cannot be less than 8 for if such were the case the postulation of C_8 alone would be too great for it to be double on a sextic surface. There remain two possible values for the genus, namely 8 and 9. The curve C_8 $(p = 8)$ is a partial intersection of a quadric and a quintic surface and has a single infinity of lines meeting it in five points, the generators of the quadric surface. Thus each sextic surface has a fixed quadric as a component and the involution is reduced to one determined by quartic surfaces. Similarly, C_8 $(p = 9)$ is the complete intersection of a quadric and a quartic surface; each sextic surface again has a fixed quadric as a component. If the order of the double curve is 7 its genus must be at least 6 for otherwise the postulation of C_7 alone is too great for it to be double on a sextic surface. The only remaining case C_7 $(p = 6)$ is a partial intersection of a quadric and a quartic surface and hence has a single infinity of quadri-secants. This quadric must be a component of each sextic surface. If the order of the double curve is 6 the genus cannot be less than 3, otherwise the postulation of C_6 alone would be too great for it to be double on a sextic surface. The curve C_6 $(p = 3)$ has equivalence and postulation of 26 and 16 respectively on cubic surfaces. The residual intersection of two such cubic surfaces is a space cubic. A birational $(3, 3)$ transformation will reduce all s_6 on which C_6 $(p = 3)$ is a double curve to quadric surfaces.

12. If the double basis curve is of order 6 and genus 4 it is necessary that $s = 2m$ and that no basis point of multiplicity greater than 2 exist. The equations of equivalence and postulation reduce to

$$4\beta_{20} + 2\beta_{21} = 10 - 2m + p$$

The following solutions exist. (There are two other geometrically consistent solutions, $m = 4$, $p = -2$ and $m = 3$, $p = -2$, $\beta_{21} = 1$. However, because of the composite nature of C_m and of the resulting surfaces ϕ, a discussion of these is outside the scope of this paper.)

	A	B	C	D	E	F	G	H	I	J
m	6	5	4	3	3	2	2	1	1	1
p	2	0	0	0	0	0	0	0	0	0
s	12	10	8	6	6	4	4	2	2	2
β_{20}	0	0	0	0	1	1	0	2	0	1
β_{21}	0	0	1	2	0	1	3	0	4	2
μ	6	7	8	9	9	10	10	11	11	11
π	2	2	2	2	2	2	2	2	2	2

The curve C_6 determines a quadric surface D_2 which corresponds to a point

in (x'). In I each point of D_2 corresponds to the whole surface. The characteristics of the above cases are alike; they will be treated as a group using m as parameter.

$$s_1 \sim s'_{12-m} \sim s_1 + D_2^2 + s_{67-6m} : C_6^{22-2m} + C_m^{12-m} + \text{parasitic curves.}$$
$$L'_6 \sim D_2 + K_{16} : C_6^5 + C_m^3; \quad J_{20} \sim D_2^2 + K_{16}$$
$$C_6 \sim S'_{24-2m} \sim D_2^6 + S_{132-12m} : C_6^{43-4m} + C_m^{24-2m}.$$
$$C_m \sim T'_{22-2m} \sim T_{132-2m} : C_6^{44-4m} + C_m^{22-2m}.$$

The number of the various parasitic curves may be obtained from a consideration of the following relations. The double curve g' of s' is of order $\frac{m^2 - 21m + 102}{2}$ and its image curve g is of order $51 - 6m$.

$$a_1 + 8a_2 + 27a_3 + 64a_4 + 125a_5 + \cdots = 1518 - 414m + 36m^2 - m^3$$
$$a_1 + 4a_2 + 9a_3 + 16a_4 + 25a_5 + \cdots = 395 - 72m + 3m^2$$
$$\eta + 2a_2 + 6a_3 + 12a_4 + 20a_5 + \cdots = 306 - 63m + 3m^3$$

The characteristics of any particular case may be obtained by substituting the proper value of m in the above relations.

13. It is impossible to have a curve of order 5 and genus less than 1 double on sextic surfaces for the postulation is too great. The curve C_5 $(p = 1)$ has equivalence and postulation of 25 and 15 respectively on cubic surfaces. Choose a point P, either an isolated basis point or a point of the simple basis curve, making the combined equivalence and postulation of C_5 and P 26 and 16 respectively. A birational transformation will reduce the sextics of web ϕ to cubic surfaces.

The curve C_5 $(p = 2)$ has a quadric surface R_2 as surface of trisecants. Each generator of R_2 corresponds to a point in (x'); R_2 as a whole corresponds to a curve r'_2. The variable intersection of two surfaces ϕ cannot meet R_2 in a point other than a point of C_5 for such would necessitate that all lines in (x') meet r'_2. Since C_5 lies on a quadric surface, C_m can meet it in not more than $2m$ points. It follows that $s = 2m - k$ where k is zero or a positive integer. But if k has a value other than zero all cases that may arise are but special types of cases in which k equals zero, for through each point in which C_m meets R_2 and not C_5 there is a generator of R_2 which is basis on ϕ. The proper C_m plus this line (or these lines) make up a new simple basis curve for which k equals zero. Thus there is no restriction in making $s = 2m$. For $\pi = 2$ there can be no simple basis point. It follows from the equations of equivalence and postulation that there can be no point of mul-

tiplicity greater than 2 on ϕ. The equations of equivalence and postulation reduce to

$$4\beta_{20} + 2\beta_{21} = 14 - 2m + p.$$

Before enumerating the solutions of this equation it is possible to discard a great number as leading either to a geometrical contradiction or to cases which are reducible. From the postulation of C_5 plus C_m it is found that if $m + 4\beta_{20} + 2\beta_{21} \leq 11$ a pencil of cubic surfaces will go through C_5 and C_m. However, this is contradictory if m exceeds 4. Other cases may be discarded because a cubic surface will contain C_5 and C_m while a web or more of cubic surfaces goes through the remaining basis elements. There is no need to consider any case where C_m is composite in such a way that one component is a line meeting C_5 twice. All such cases are reducible by a $(3, 3)$ birational transformation whose basis elements are C_5 and this line. The remaining solutions are given in the table.

	A	B	C	D	E	F	G	H
m	9	8	7	6	7	6	5	4
p	4	2	0	-2	2	0	0	0
s	18	16	14	12	14	12	10	8
β_{10}	0	0	0	0	0	0	0	0
$2\beta_{20} + \beta_{21}$	0	0	0	0	1	1	2	3
μ	7	8	9	10	9	10	11	12
π	2	2	2	2	2	2	2	2

The curve r'_2 corresponds to R_2 plus a curve r of order $25 - 2m$. This order may be obtained as previously explained or by considering the non-basis intersection of an s_4 and an s_{10}, images respectively of a plane and a quadric containing r'_2. Besides basis elements K_{18} and R_2 have in common six generators $6t$ of the latter; these correspond to the six contacts of r'_2 and L'_6. Using m as parameter the correspondence is

$$s_1 \sim s_{16-m} \sim s_1 + R_2 + s_{93-6m} : C_5^{31-2m} + C_m^{16-m} + r + 6t.$$

$$L'_6 \sim K_{18} : C_5^6 + C_m^3; \quad J_{20} \equiv K_{18} + R_2.$$

$$C_5 \sim S'_{32-2m} \sim R^3 + S_{186-12m} : C_5^{62-4m} + C_m^{32-2m}.$$

$$C_m \sim T'_{30-2m} \sim T_{180-12m} : C_5^{60-4m} + C_m^{31-2m}.$$

The number of the various parasitic curves may be determined by use of the following relations. The curve g is of order $75 - 6m$ and its image curve g' is of order $\dfrac{m^2 - 29m + 200}{2}$.

$$p_g = \frac{2402 - 548m + 41m^2 - m^3}{2} - 3a_3 - 12a_4 - 30a_5 - 60a_6 \cdots$$

$$p_{g'} = \frac{4476 - 1057m + 81m^2 - 2m^3}{6} - 2a_3 - 8a_4 - 20a_5 - 40a_6 \cdots$$

$$\eta = \frac{m^3 - 39m^2 + 470m - 1740}{3} + 2a_3 + 8a_4 + 20a_5 + 40a_6 \cdots$$

$$a_1 + 8a_2 + 27a_3 + 64a_4 + 125a_5 + \cdots = 3720 - 744m + 48m^2 - m^3$$

$$a_1 + 4a_2 + 9a_3 + 16a_4 + 25a_5 + \cdots = 720 - 96m + 3m^2$$

$$2a_2 + 8a_3 + 20a_4 + 40a_5 + \cdots = \frac{3540 - 731m + 48m^2 - m^3}{3}$$

For each of the above cases one can substitute in these equations the proper value for m and obtain the details for that particular case.

14. When the double curve is a quartic it is impossible that its genus be -3 for then C_μ would be composite. If the genus is -2, C_4 must consist of a conic and two lines mutually skew. A line in the plane of the conic meeting both the double basis lines is itself basis. The conic and these three lines form the basis elements of a $(3, 4)$ birational transformation by means of which ϕ may be reduced to a web of quintics. The curve C_4 $(p = -1)$ together with two points furnish the basis elements for a $(3, 5)$ birational transformation which reduces the web of sextic surfaces to a web of quartic surfaces. If C_4 is rational web ϕ is reducible provided there exists a basis point of multiplicity three or more on ϕ but not a point of C_4. A line may be drawn through this basis point which will meet C_4 twice and be therefore itself basis. A birational transformation whose basis elements are C_4, this line and a basis point reduces the sextics to surfaces of lower order. This same transformation works a reduction, even if there is no triple basis point, if C_m is composite in such a way that one component of it is a line which meets C_4 twice. For this reason it is necessary to impose the restriction that $m \geq 2 - 2p$. As C_4 $(p = 0)$ lies on a quadric surface and as C_m cannot be on the same quadric $s \leq 2m$. As previously explained it is no restriction to make $s = 2m$. There can be no simple basis point when $\pi = 2$; from substituting this fact in the equations of equivalence and postulation it follows that no point of C_4 can be of greater multiplicity on ϕ than is the curve as a whole. If $m + 4\beta_{20} + 2\beta_{21} \geq 14$ a pencil of cubics can go through C_4 and C_m but this is contradictory if m exceeds 5. The only remaining cases are given in the table below.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
m	13	12	11	10	9	8	11	10	9	8	7	9	8	7	6	7	6	5	5	4	3	2
p	8	6	4	2	0	−2	6	4	2	0	−2	4	2	0	−2	2	0	0	2	0	0	0

s	26	24	22	20	18	16	22	20	18	16	14	18	16	14	12	14	12	10	10	8	6	4
$2\beta_{20}+\beta_{21}$	0	0	0	0	0	0	1	1	1	1	1	2	2	2	2	3	3	4	5	5	6	7
μ	7	8	9	10	11	12	9	10	11	12	13	11	12	13	14	13	14	15	15	16	17	18
π	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

In all the above cases the correspondence follows the same law. The curve C_4 $(p=0)$ determines a quadric surface R_2 which has for image in (x') a quartic curve r'_4 and in I a curve r of order $66-4m$. The correspondence is

$$L'_6 \sim K_{18} : C_4^6 + C_m^3; \quad J_{20} \equiv K_{18} + R_2; \quad (K_{18}, R_2) = 12t.$$
$$s_1 \sim s'_{20-m} \sim s_1 + R_2 + s_{117-6m} \cdot \quad C_4^{39-2m} + C_m^{20-m}.$$
$$C_4 \sim S'_{40-2m} \sim R_2^3 + S_{234-12m} : C_4^{78-4m} + C_m^{40-2m}.$$
$$C_m \sim T'_{38-2m} \sim T_{228-12m} : C_4^{76-4m} + C_m^{40-2m}.$$

Suppose C_4 is of genus 1. If there is a point of multiplicity greater than 2 on ϕ which is not a point of C_4 the web of sextics can be reduced birationally by a $(3,4)$ transformation. The basis elements of this transformation are C_4, any line meeting C_4 once and the triple point. For $\pi=3$ it is necessary that $\beta_{10}=0$ and from the equations of equivalence and postulation it follows that $\beta_{j2}=0$ for all j. If there exists a double basis point, C_4 plus this point determines a quadric surface while a web of quartic surfaces goes through the remaining basis elements. In order that each s_6 shall not have as a component a fixed cubic surface through C_4 and C_m it is necessary that m be less than 11. The remaining cases are given in the table. $(\mu = 20-m,\ \pi=3)$

	A	B	C	D	E	F
m	10	9	8	7	6	5
p	9	7	5	3	1	-1
s	14	12	10	8	6	4

A quartic surface D_4 is fixed by C_4 and C_m while a pencil of quadric surfaces goes through C_4 alone. The image of D_4 in (x') is a line d' and in I it is self-corresponding. Each point of d' corresponds to a curve of order $16-m$ on D_4. The correspondence is

$$L'_8 \sim K_{16} : C_4^6 + C_m^2 + D_4^2; \quad J_{20} \equiv K_{16} + D_4.$$
$$s_1 \sim s_{20-m} \sim s_1 + D_4^{16-m} + s_{55-2m} : C_4^{24-m} + C_m^4.$$
$$C_4 \sim S'_{38-2m} \sim D_4^{30-2m} + S_{108-4m} : C_4^{47-2m} + C_m^8.$$
$$C_m \sim T'_{42-2m} \sim D_4^{36-2m} + T_{108-4m} : C_4^{48-2m} + C_m^7.$$

There is no parasitic curve of order greater than 3. The number of fundamental lines, conics and cubics respectively is $27+m$, $36-2m$ and $13-m$.

15. Suppose the double curve is a cubic. If the genus is -2, C_3 determines a quadric surface R_2. As previously shown it is no restriction to make $s = 2m$. The postulation is such that if there is a basis point of multiplicity greater than 2, R_2 is a fixed component of each s_6. From the equations of equivalence and postulation one finds that $\beta_{10} = 0$ and $p = 2m - 22 + 4\beta_{20} + 2\beta_{21}$. The correspondence in all cases that arise follows the same law. The general characteristics are

$$L'_6 \sim K_{18} : C_3^6 + C_m^3 ; \quad J_{20} \equiv K_{18} + R_2 ; \quad (K_{18}, R_2) = 18t.$$
$$s_1 \sim s'_{24-m} \sim s_1 + R_2 + s_{141-6m} : \quad C_3^{47-2m} + C_m^{24-m}.$$
$$R_2 \sim r'_6 \sim R_2 + r_{123-6m}.$$
$$C_3 \sim S'_{48-2m} \sim R_2^3 + S_{282-12m} ; \quad C_m \sim T'_{46-2m} \sim T_{276-12m}.$$

Suppose the double basis curve is of order 3 and genus -1, that is consists of a conic and a line skew to the conic. If there is a basis point of multiplicity 3 or more, which is not a point of the conic, the surfaces are reducible. First, if the triple point is in the plane of the conic this plane is a component of each s_6. Second, if the triple point is not in the plane of the conic a $(2, 2)$ birational transformation will reduce ϕ to a web of quintic surfaces. The double conic and the triple point each taken simple are the basis elements of this transformation. The equations of equivalence and postulation have no consistent solution when a point of the conic is triple or quadruple on ϕ. There remain therefore only basis points whose multiplicities on ϕ do not exceed 2. The genus of C_μ is 3. It is necessary that there exists a surface D_4. This implies that the associated web of quadrics have in common a curve of order 4 and genus 1 (section 4). This is impossible here.

If the double cubic be of genus 0 no I has been found to exist. Contradictory relations arise with regard to parasitic surfaces. There is, however, an involution J of order 2 associated with every pencil ψ of web ϕ. If a point P_1 is fixed it determines a surface ψ_1 while the line through P_1 and meeting C_3 $(p = 0)$ twice meets ψ_1 in but one other point Q_1. The involution J is between such pairs of points as P_1 and Q_1. The congruence of lines which meets C_3 twice is birationally equivalent to a bundle of lines through some point A. Thus J can be birationally reduced to a monoidal involution and as such has already been studied.

When the double cubic has genus 1 no involutorial correspondence has been found.

16. It will save repetition to make a digression here. In the last section no I could be found when C_3 was of genus 0 or -1. The reason given

was that contradictory relations existed with respect to parasitic surfaces. The same difficulty enters in all cases yet to be considered. Specifically the complication is that π is so great and therefore the orders of L' and $\bar{K}$ so large that no parasitic surface or surfaces D can be found which satisfy a consistent relation among $\bar{K}$, K, D and J. It is needful here to restate the conditions necessary for the existence of a surface D.

If a surface D corresponds to a line d' there must exist a pencil of surfaces S which corresponds to the pencil of planes through d' and are such that D together with an S form a sextic of web ϕ. Two of the planes in (x') meet in no point other than d'; it follows that the two corresponding surfaces S meet either in basis curves alone or contain besides basis curves some fixed curve on D. There exist therefore three relations among the curves through which each S goes, namely, (1) their combined postulation must be such that there is a pencil of surfaces S; (2) their combined equivalence must be such that three S meet in no variable point; (3) two S can meet in no variable curve.

If D corresponds to a point D' the above relations become (1) the combined postulation of the curves through which S goes must be such that there is a net of surfaces S; (2) their combined equivalence must be such that three surfaces S meet in no variable point; (3) two S meet in a variable curve of order $i^2 - j$ where S is of order i and j is the order of the image of a line through D'.

Whenever a case is excluded because of contradictory relations with respect to parasitic surfaces it means that no D can be found that will satisfy the necessary conditions (1) among $\bar{K}$, K, D and J, and (2) as to containing the necessary basis elements.

17. Suppose the double curve is a proper conic and consider the following cases. (1) If there is a basis point of multiplicity greater than 2 which is not in the plane of the conic a $(2, 2)$ birational transformation will reduce web ϕ. (2) If there is a basis point of multiplicity greater than 2 in the plane of the conic but not a point of the conic this plane is a component of each ϕ. (3) In all other cases a quartic surface will contain the conic single fold, the simple curve and all the basis points while more than a web of quadric surfaces will go through the conic alone.

Suppose the double basis curve consists of two skew lines. If there are three basis points whose combined multiplicities equal or exceed 9 any I that might exist is reducible by a $(2, 3)$ birational transformation whose basis elements are one of the double lines and the three points. If there is a quadruple basis point all solutions of the equations of equivalence and postulation are such that the simple curve must meet the plane of this point and one

of the double lines in at least two points; thus any I that might result may either be generated by surfaces of order lower than 6 or disintegrates. In all other cases the value of π is so great as to introduce inconsistent relations among the parasitic surfaces. As with C_3 $(p = 0)$ there exists in this case an involution J defined in the same way.

Assume that the double curve is a straight line. In case there are three basis points whose combined multiplicities equal at least 9 the $(2, 3)$ transformation of the preceding paragraph again applies. In all other cases no parasitic surfaces exist which satisfy the necessary conditions.

If one sets up the equations of equivalence and postulation so as to have only a simple basis curve together with basis points or to have basis points alone no I results.

18. Although it was shown that involutions exist in cases where there are double basis points among the basis elements no account was taken of this fact in tabulating the previous correspondences. The entire revision can now be conveniently effected. An isolated double point has for image in (x') a quartic surface which returns to (x) as a surface of order 24; a point of a simple basis curve which is double on all surfaces ϕ has for image in (x') a cubic surface which returns to (x) as a surface of order 18. These surfaces may or may not have parasitic surfaces as components. If there is a double basis point among the basis elements of ϕ, C_μ meets some or all the basis curves in a less number of points than if no double basis point existed. Thus the orders of $S, T, \cdots$ are accordingly decreased. The order of the jacobian J of web s_N is independent of the number of double basis points and if there is no such point is made up of $S, T, \cdots$ and R. If there is a double basis point the image in I of this point is included in J.

III. CONCLUSION.

19. This study has attempted to classify but a limited number of the $(2, 1)$ correspondences associated with a web of sextic surfaces. Within the bounds set in the Introduction, it is believed that this classification is complete. The work done here has also suggested avenues for future investigation, for example, a more detailed discussion of certain irregular cases, or, an exposition of the properties of the various types of basic and parasitic elements.

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